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MARGINAL STANDARDIZATION OF UPPER SEMICONTINUOUS PROCESSES. WITH APPLICATION TO MAX-STABLE PROCESSES

ANNE SABOURIN AND JOHAN SEGERS

Abstract. In the field of spatial extremes, stochastic processes with upper semicontinuous (usc) trajectories have been proposed as random shape functions for max-stable models. In the literature dealing with usc processes, max-stability is defined via a sequences of scaling constants, rather than functions, only. It is however not clear whether and how extreme-value theory (EVT) for continuous processes extends to usc processes. In particular, classical multivariate and continuous EVT relies on the probability integral transform and Sklar's theorem. This theorem justifies working with standard marginal distributions, simplifying the task of constructing and characterizing max-stable processes and their domains of attraction. In the present work, we investigate the possibility to follow these steps for usc processes. Unfortunately, the pointwise probability integral transform is not necessarily 'permitted': without additional assumptions, the obtained process may not even have usc trajectories. We give sufficient conditions for marginal standardization to be possible, and we state a partial extension of Sklar's theorem for usc processes, with a particular focus on max-stable ones.

Key words. extreme-value theory; max-stable processes; semicontinuous processes; copulas.

1. Introduction

Max-stable processes and generalized Pareto processes appear respectively as the weak limits of normalized pointwise maxima and high-threshold excesses, respectively, of stochastic processes. This motivates their use in geostatistics to model spatial processes at extreme levels. Stochastic process models with semicontinuous trajectories arise naturally when extremes are localized in space. Think of the behaviour of rain and wind at two sides of a mountain ridge. Another example is a rainstorm model with storms being supported by random closed patches, outside of which there is no rain. Various semicontinuous max-stable models have been proposed in the literature (Schlather, 2002; Davison and Gholamrezaee, 2012; Huser and Davison, 2014). Since our focus is on upper extremes, we will henceforth work with stochastic processes with upper semicontinuous (usc) trajectories.

Consider a stochastic process, $\xi = (\xi(s))_{s\in\mathbb{D}}$. We will refer to s as the 'space variable'. Throughout this paper, the index set $\mathbb{D}$ is a non-empty, compact subset of some finite-dimensional Euclidean space with no isolated points. One way to

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deal with upper semicontinuity is to consider ξ as a collection of random variables of which the trajectories are usc almost surely. The drawback of such an approach is that the finite-dimensional distributions do not give a handle on path-wise properties. Think for instance of the supremum of a trajectory over a subregion of $\mathbb{D}$.

To handle such path functionals of ξ, it is convenient to consider it as a random element in some function space. For functions which are not necessarily continuous, Skorohod spaces may come to mind. These are less natural, however, when the dimension of the domain $\mathbb{D}$ is higher than one and the meaning of 'left' and 'right' depends on the choice of the coordinate system. In this paper, we follow Norberg (1987) and Resnick and Roy (1991) and work with the space $\mathrm{USC}(\mathbb{D})$ of upper semicontinuous functions on $\mathbb{D}$ equipped with the hypo-topology. We will assume that ξ is a *usc process*, i.e., a Borel measurable map from some probability space into $\mathrm{USC}(\mathbb{D})$; see Section 2 for details.

A common approach in multivariate and spatial analysis is to split the analysis and modelling of ξ into two parts: the marginal distributions and the 'dependence structure'. By the latter is meant the distribution of the random object ξ^* that arises from ξ by standardizing the random variables $\xi(s)$ to a common univariate distribution. In Sklar's theorem, this is the uniform distribution on $[0, 1]$, whereas for max-stable processes, the unit-Fréchet distribution is most common.

For usc processes, it is not at all clear from the outset whether this is possible too. Does the pointwise probability integral transform turns a given usc process into another one? If yes, is this transformation a measurable one on $\mathrm{USC}(\mathbb{D})$? And can the original usc process be reconstructed from the standardized version and the marginal distributions?

We will provide answers to these questions for general usc processes and then specialize these to max-stable ones. Some side conditions will enter, and we will show that unexpected things go wrong if these are not met.

Compared to the set-up with processes with continuous trajectories, the situation is complicated by two issues. First, the pointwise transformations need not be continuous with respect to $s \in \mathbb{D}$, so that it is not clear that upper semicontinuity of the trajectories is preserved. Second, the Borel σ-field on $\mathrm{USC}(\mathbb{D})$ is larger than the one generated by finite-dimensional cylinders. Not only does this pose challenges to proving measurability of certain transformations, moreover, the law of a usc process is not determined by its finite-dimensional distributions.

Max-infinitely divisible usc processes have been considered in Giné et al. (1990), and extremes of usc processes have been investigated, among others, by Norberg (1987) and Resnick and Roy (1991). Max-stability is then defined via a sequence of norming *constants*, which are the same for every spatial location. Such processes are closely related to *union-stable* random sets (Molchanov, 2005, Chapter 4). The norming constants involved in the definition of max-stability determine the shape parameter of the marginal distributions. Imposing a sequence of norming *constants* rather than functions then forces the margins of the usc process to be of the same type everywhere. However, modeling spatial extremes often requires space-varying marginal parameters. This situation is not covered by the present literature, and motivates the contribution of our paper.

Section 2 sets up the necessary background about usc processes. A general class of measurable transformations on the space of usc functions is introduced in Section 3. Under regularity conditions on the marginal distributions, this class includes the pointwise probability integral transform and its inverse. This property allows to state a partial generalization of Sklar's theorem for usc processes in general (Section 4) and for max-stable ones in particular (Section 5). Section 6 concludes. Some additional results are deferred to the appendices.

2. Random usc processes

We review some essential definitions and properties of random usc processes, or usc processes for short. The material in this section may for instance be found in Salinetti and Wets (1986), Beer (1993, Chapter 5), and Molchanov (2005, Chapter 1.1 and Appendix B).

Recall that $\mathbb{D}$ is a non-empty, compact subset of some finite-dimensional Euclidean space with no isolated points. A function $x : \mathbb{D} \to [-\infty, \infty]$ is *upper semicontinuous* (usc) if and only if the set $\{s \in \mathbb{D} : x(s) \geq \alpha\}$ is closed for each $\alpha \in \mathbb{R}$. The definition of *lower semicontinuous* (lsc) functions is similar, the inequality being reversed.

For $x : \mathbb{D} \to [-\infty, +\infty]$, define the *epigraph* and *hypograph* by

$$\operatorname{epi} x = \{(s, \alpha) \in \mathbb{D} \times \mathbb{R} : x(s) \leq \alpha\},$$
$$\operatorname{hypo} x = \{(s, \alpha) \in \mathbb{D} \times \mathbb{R} : \alpha \leq x(s)\}.$$

Observe that $\operatorname{epi} x$ and $\operatorname{hypo} x$ are subsets of $\mathbb{D} \times \mathbb{R}$, even if the range of x contains $-\infty$ or $+\infty$. A function is upper or lower semicontinuous if and only if its hypograph or epigraph, respectively, is closed.

Let $\mathrm{USC}(\mathbb{D})$ be the collection of all upper semicontinuous functions from $\mathbb{D}$ into $[-\infty, \infty]$. By identifying the function $z \in \mathrm{USC}(\mathbb{D})$ with the set $\operatorname{hypo} z \subset \mathbb{D} \times \mathbb{R}$, any topology on the space $\mathcal{F} = \mathcal{F}(\mathbb{D} \times \mathbb{R})$ of closed subsets of $\mathbb{D} \times \mathbb{R}$ results in a trace topology on the space of usc functions.

The Fell *hit-and-miss* topology on $\mathcal{F}$ is defined as follows. Let $\mathcal{K}$ and $\mathcal{G}$ denote the families of compact and open subsets of $\mathbb{D} \times \mathbb{R}$, respectively. A base for the Fell topology on $\mathcal{F}$ is the family of sets of the form

$$\mathcal{F}^K_{G_1,\ldots,G_n} = \{F \in \mathcal{F} \;:\; F \cap K = \varnothing,\, F \cap G_1 \neq \varnothing,\, \ldots,\, F \cap G_n \neq \varnothing\}$$

for $K \in \mathcal{K}$ and $G_1, \ldots, G_n \in \mathcal{G}$. A net of closed sets converges to a limit set F if and only if every compact K missed by F is eventually also missed by the net and if every open G hit by F is eventually also hit by the net.

The Fell topology on $\mathcal{F}(\mathbb{D} \times \mathbb{R})$ induces a trace topology on $\mathrm{USC}(\mathbb{D})$, the *hypo-topology*. A sequence x_n in $\mathrm{USC}(\mathbb{D})$ is said to *hypo-converge* to x in $\mathrm{USC}(\mathbb{D})$ if and only if $\operatorname{hypo} x_n$ converges to $\operatorname{hypo} x$ in the Fell topology. Since the underlying space $\mathbb{D}\times\mathbb{R}$ is locally compact, Hausdorff, and second countable (LCHS), the space $\mathrm{USC}(\mathbb{D})$ thus becomes a compact, Hausdorff, second-countable space. A convenient pointwise criterion for hypo-convergence is the following one: a sequence $x_n \in \mathrm{USC}(\mathbb{D})$ hypo-converges to $x \in \mathrm{USC}(\mathbb{D})$ if, and only if,

$$\forall s \in \mathbb{D} : \forall s_n \in \mathbb{D},\ s_n \to s : \limsup_{n\to\infty} x_n(s_n) \le x(s);$$
$$\forall s \in \mathbb{D} : \exists s_n \in \mathbb{D},\ s_n \to s : \liminf_{n\to\infty} x_n(s_n) \ge x(s).$$

Let $(\Omega, \mathcal{A}, \Pr)$ be a complete probability space and let $\mathcal{B}(\mathcal{F})$ be the Borel σ-field on $\mathcal{F} = \mathcal{F}(\mathbb{D}\times\mathbb{R})$ generated by the Fell topology. A random closed set of $\mathbb{D}\times\mathbb{R}$ is a Borel measurable map from Ω into $\mathcal{F}$. Similarly, a usc process ξ on $\mathbb{D}$ is a Borel measurable map from Ω into $\mathrm{USC}(\mathbb{D})$. Equivalently, $\operatorname{hypo}\xi$ is a random closed subset of $\mathbb{D}\times\mathbb{R}$ taking values in the collection of all closed hypographs. Some authors use the term 'normal integrand' to refer to such processes, as a special case of stochastic processes with usc realizations (Salinetti and Wets, 1986, p. 12). In this work, a usc process is always a function $\xi : \mathbb{D}\times\Omega \to [-\infty,\infty]$ such that the map $\Omega \to \mathcal{F} : \omega \mapsto \operatorname{hypo}\xi(\,\cdot\,,\omega)$ is Borel measurable.

The *capacity functional* of $\operatorname{hypo}\xi$ is the map $T : \mathcal{K} \to [0,1]$ defined by

$$T(K) = \Pr\{(\operatorname{hypo}\xi)\cap K \neq \varnothing\}, \qquad K \in \mathcal{K}.$$

We also call T the capacity functional of ξ rather than of $\operatorname{hypo}\xi$. Since $\mathbb{D}\times\mathbb{R}$ is LCHS, the Borel σ-field $\mathcal{B}(\mathcal{F})$ is generated by the sets

$$\mathcal{F}_K = \{F \in \mathcal{F} : F \cap K \neq \varnothing\}, \qquad K \in \mathcal{K}.$$

(Molchanov, 2005, p. 2). The collection of complements of the sets $\mathcal{F}_K$ being a π-system, the capacity functional determines the distribution of a random closed set or a usc process.

We emphasize that the Borel σ-field on $\mathrm{USC}(\mathbb{D})$ is strictly larger than the one generated by the finite-dimensional cylinders. Without additional hypotheses, the finite-dimensional distributions of a usc process do not determine its distribution as a usc process. The evaluation mappings $\mathrm{USC}(\mathbb{D}) \to [-\infty,\infty] : x \mapsto x(s)$ being hypo-measurable, a usc process ξ is also a stochastic process (i.e., a collection of random variables) with usc trajectories. The converse is not true, however: for such stochastic processes, the map $\Omega \to \mathrm{USC}(\mathbb{D}) : \omega \mapsto \xi(\,\cdot\,,\omega)$ is not necessarily hypo-measurable.

3. Transformations of usc functions

The following class of functions plays a fundamental role in our analysis of the pointwise probability integral transform and its inverse.

Definition 3.1. A function $U : \mathbb{D}\times[-\infty,\infty] \to [-\infty,\infty]$ belongs to the class $\mathcal{U}(\mathbb{D})$ if it has the following two properties:

(a) For every $s \in \mathbb{D}$, the map $x \mapsto U(s,x)$ is non-decreasing and right-continuous.
(b) For every $x \in \mathbb{R} \cup \{\infty\}$, the map $s \mapsto U(s,x)$ is usc.

See Remark 3.1 for the reason why we did not include $x = -\infty$ in condition (b).

Proposition 3.1. *Let $U \in \mathcal{U}(\mathbb{D})$.*

(i) If z_n hypo-converges to z in $\mathrm{USC}(\mathbb{D})$ *and if $s_n \to s$ in $\mathbb{D}$ as $n \to \infty$, then* $\limsup_{n\to\infty} U(s_n, z_n(s_n)) \le U(s, z(s))$.
(ii) For every $z \in \mathrm{USC}(\mathbb{D})$, the function $U_*(z) : \mathbb{D} \to [-\infty, \infty]$ defined by $(U_*(z))(s) = U(s, z(s))$ belongs to $\mathrm{USC}(\mathbb{D})$ *too.*
(iii) For every compact $K \subset \mathbb{D} \times \mathbb{R}$, the set $\{z \in \mathrm{USC}(\mathbb{D}) : \operatorname{hypo} U_*(z) \cap K \neq \varnothing\}$ is hypo-closed.
(iv) The map $U_* : \mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ is hypo-measurable.

Proof. *(i)* Consider two cases: $z(s) = \infty$ and $z(s) < \infty$.

Suppose first that $z(s) = \infty$. By the assumptions on U, we have

$$\limsup_{n\to\infty} U(s_n, z_n(s_n)) \le \limsup_{n\to\infty} U(s_n, \infty) \le U(s, \infty) = U(s, z(s)).$$

Suppose next that $z(s) < \infty$. Let $y > z(s)$. By hypo-convergence, we have $\limsup_{n\to\infty} z_n(s_n) \le z(s)$. As a consequence, there exists a positive integer $n(y)$ such that $z_n(s_n) \le y$ for all $n \ge n(y)$. By the properties of U, we have

$$\limsup_{n\to\infty} U(s_n, z_n(s_n)) \le \limsup_{n\to\infty} U(s_n, y) \le U(s, y).$$

Since $y > z(s)$ was arbitrary and since $x \mapsto U(s,x)$ is non-decreasing and right-continuous, we find $\limsup_{n\to\infty} U(s_n, z_n(s_n)) \le \inf_{y > z(s)} U(s,y) = U(s, z(s))$.

(ii) In statement (i), set $z_n = z$ to see that $\limsup_{n\to\infty} U(s_n, z(s_n)) \le U(s, z(s))$ whenever $s_n \to s$ in $\mathbb{D}$ as $n \to \infty$.

(iii) Let $K \subset \mathbb{D} \times \mathbb{R}$ be compact. Suppose that z_n hypo-converges to z in $\mathrm{USC}(\mathbb{D})$ as $n \to \infty$ and that $\operatorname{hypo} U_*(z_n) \cap K \neq \varnothing$ for all positive integer n. We need to prove that $\operatorname{hypo} U_*(z) \cap K \neq \varnothing$ too.

For each positive integer n, there exists $(s_n, x_n) \in K$ such that $U(s_n, z_n(s_n)) \ge x_n$. Since K is compact, we can find a subsequence $n(k)$ such that $(s_{n(k)}, x_{n(k)}) \to (s, x) \in K$ as $k \to \infty$. By (i), we have

$$U(s, z(s)) \ge \limsup_{k\to\infty} U(s_{n(k)}, z_{n(k)}(s_{n(k)})) \ge \limsup_{k\to\infty} x_{n(k)} = x.$$

As a consequence, $(s, x) \in \operatorname{hypo} U_*(z) \cap K$.

(iv) The collection of sets of the form $\{z \in \mathrm{USC}(\mathbb{D}) : \operatorname{hypo}(z) \cap K \neq \varnothing\}$, where K ranges over the compact subsets of $\mathbb{D} \times \mathbb{R}$, generates the Borel σ-field on $\mathrm{USC}(\mathbb{D})$ induced by the hypo-topology. By (iii), the inverse image under U_* of each such set is hypo-closed and thus hypo-measurable. We conclude that U_* is hypo-measurable. □

Remark 3.1. In part (b) of the definition of $\mathcal{U}(\mathbb{D})$, we did not include $x = -\infty$. The reason is that this case is automatically included: by Proposition 3.1(ii) applied to the function $z(s) \equiv -\infty$, the map $s \mapsto U(s, -\infty)$ is necessarily usc too.

Condition (b) is also necessary for the conclusion of Proposition 3.1(ii) to hold: given x, define $z(s) \equiv x$.

A convenient property of $\mathcal{U}(\mathbb{D})$ is that it is closed under an appropriate kind of composition. This allows to deconstruct complicated transformations in terms of more elementary ones.

Lemma 3.1. *If U and V belong to $\mathcal{U}(\mathbb{D})$, the function W defined by $(s, x) \mapsto V(s, U(s, x))$ belongs to $\mathcal{U}(\mathbb{D})$ too, and $W_* = V_* \circ U_*$.*

Proof. First, fix $s \in \mathbb{D}$. The map $x \mapsto V(s, U(s, x))$ is non-decreasing: for $-\infty \leq x \leq y \leq \infty$, we have $U(s, x) \leq U(s, y)$ and thus $V(s, U(s, x)) \leq V(s, U(s, y))$. The map $x \mapsto V(s, U(s, x))$ is also right-continuous: if x_n converges from the right to x as $n \to \infty$, then so does $u_n = U(s, x_n)$ to $u = U(s, x)$ and thus $V(s, u_n)$ to $V(s, u)$.

Next, fix $x \in \mathbb{R} \cup \{\infty\}$. We need to show that the function $s \mapsto V(s, U(s, x))$ is usc. But the function z defined by $s \mapsto U(s, x)$ is usc, and, by Proposition 3.1, so is the function $s \mapsto V(s, z(s)) = V(s, U(s, x))$. □

Example 3.1. Here are some simple examples of functions U in the class $\mathcal{U}(\mathbb{D})$ and the associated mappings $U_* : \mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$.

(i) If $f : [-\infty, \infty] \to [-\infty, \infty]$ is non-decreasing and right-continuous, the function $(s, x) \mapsto f(x)$ belongs to $\mathcal{U}(\mathbb{D})$. The associated map $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ is $z \mapsto f \circ z$.

(ii) If $y \in \mathrm{USC}(\mathbb{D})$, the functions $(s, x) \mapsto x \vee y(s)$ and $(s, x) \mapsto x \wedge y(s)$ both belong to $\mathcal{U}(\mathbb{D})$. The associated maps $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ are $z \mapsto z \vee y$ and $z \mapsto z \wedge y$, respectively.

(iii) If $a : \mathbb{D} \to (0, \infty)$ is continuous, then the function $(s, x) \mapsto a(s)\, x$ belongs to $\mathcal{U}(\mathbb{D})$. If $b : \mathbb{D} \to \mathbb{R}$ is usc, then the map $(s, x) \mapsto x + b(x)$ belongs to $\mathcal{U}(\mathbb{D})$. The associated maps $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ are $z \mapsto az$ and $z \mapsto z + b$, respectively.

A more elaborate example of a function in $\mathcal{U}(\mathbb{D})$ is induced by the collection of right-continuous quantile functions of the marginal distributions of a stochastic process with usc trajectories. See Appendix A for some background on right-continuous quantile functions.

Lemma 3.2. *Let $\xi = (\xi(s) : s \in \mathbb{D})$ be a stochastic process indexed by $\mathbb{D}$ and with values in $[-\infty, \infty]$. Let $F_s : x \in [-\infty, \infty] \to F_s(x) = \Pr[\xi(s) \leq x]$ denote the right-continuous marginal distribution function of $\xi(s)$. Define*

$$Q_s(p) = \sup\{y \in \mathbb{R} : F_s(y) \leq p\}, \qquad (s, p) \in \mathbb{D} \times [0, 1].$$

If ξ has usc trajectories, the function $(s, x) \mapsto Q_s((x \vee 0) \wedge 1)$ belongs to $\mathcal{U}(\mathbb{D})$.

Proof. First, by Proposition A.1, the map $[0,1] \ni p \mapsto Q_s(p)$ is non-decreasing and right-continuous for every $s \in \mathbb{D}$, and hence the same is true for the map $[-\infty, \infty] \ni x \mapsto Q_s((x \vee 0) \wedge 1)$.

Second, fix $x \in [-\infty, \infty]$ and write $p = (x \vee 0) \wedge 1 \in [0,1]$. We need to show that the map $s \mapsto Q_s(p)$ is usc. Let $s_n \to s$ in $\mathbb{D}$ as $n \to \infty$; we need to show that $Q_s(p) \geq \limsup_{n\to\infty} Q_{s_n}(p)$. Let Q' be the right-continuous quantile function of the random variable $\limsup_{n\to\infty} \xi(s_n)$. The trajectories of ξ are usc, and thus $\xi(s) \geq \limsup_{n\to\infty} \xi(s_n)$, which implies $Q_s(p) \geq Q'(p)$. By Lemma A.2, we then find $Q_s(p) \geq Q'(s) \geq \limsup_{n\to\infty} Q_{s_n}(p)$, as required. □

4. Sklar's theorem for usc processes

A d-variate copula is the cumulative distribution function of a d-dimensional random vector with standard uniform margins. Sklar's celebrated theorem (Sklar, 1959) states two things:

(I) For every copula C and every vector $F_1, \ldots, F_d$ of univariate distribution functions, the function $(x_1, \ldots, x_d) \mapsto C(F_1(x_1), \ldots, F_d(x_d))$ is a d-variate distribution function with margins $F_1, \ldots, F_d$.

(II) Every d-variate distribution function F can be represented in this way.

Reformulated in terms of random vectors, the two statements read as follows:

(I) For every random vector $(U_1, \ldots, U_d)$ with uniform components and for every vector $F_1, \ldots, F_d$ of univariate distribution functions, the random vector $(Q_1(U_1), \ldots, Q_d(U_d))$ has marginal distributions $F_1, \ldots, F_d$, where Q_j is the (right- or left-continuous) quantile function corresponding to F_j.

(II) Every random vector $(X_1, \ldots, X_d)$ can be represented in this way.

We investigate up to what extent these statements hold for usc processes too. According to Proposition 4.1, the first statement remains true provided the sections $s \mapsto Q_s(p)$ of the right-continuous quantile functions are usc. According to Proposition 4.2, the Sklar representation is valid for usc processes whose marginal distribution functions have usc sections, and even then, the equality in distribution is not guaranteed.

Proposition 4.1 (à la Sklar I for usc processes)**.** *Let Z be a usc process having standard uniform margins. Let $(F_s : s \in \mathbb{D})$ be a family of (right-continuous) distribution functions and let $Q_s(p) = \sup\{x \in \mathbb{R} : F_s(x) \leq p\}$ for all $(s,p) \in \mathbb{D} \times [0,1]$. Define a stochastic process ξ by $\xi(s) = Q_s((Z(s) \vee 0) \wedge 1)$ for $s \in \mathbb{D}$. Then the following two statements are equivalent:*

(i) ξ is a usc process with marginal distributions given by F_s.

(ii) For every $p \in [0,1]$, the function $s \mapsto Q_s(p)$ is usc.

Proof. For every $s \in \mathbb{D}$, the random variable $(Z(s) \vee 0) \wedge 1$ is equal almost surely to $Z(s)$, so that its distribution is uniform on $[0,1]$ too. By Proposition A.1(iv), the distribution function of $\xi(s)$ is given by F_s. As a consequence, Q_s is the right-continuous quantile function of the random variable $\xi(s)$.

If (i) holds, then the trajectories $s \mapsto \xi(s)$ are usc. By Lemma 3.2, the function $s \mapsto Q_s(p)$ is usc for every $p \in [0,1]$.

Conversely, if (ii) holds, then the function U defined by $(s,x) \mapsto Q_s((x \vee 0) \wedge 1)$ belongs to $\mathcal{U}(\mathbb{D})$ defined in Definition 3.1. Let U_* be the associated map $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$; see Proposition 3.1. Then $\xi = U_*(Z)$ is a usc process since Z is a usc process and U_* is hypo-measurable by Proposition 3.1(iv). □

Proposition 4.2 (à la Sklar II for usc processes). *Let ξ be a usc process. Let $F_s(x) = \Pr[\xi(s) \le x]$ for $x \in [-\infty, \infty]$ and let $Q_s(p) = \sup\{x \in \mathbb{R} : F_s(x) \le p\}$ for $p \in [0,1]$. Suppose the following two conditions hold:*
(a) For every $s \in \mathbb{D}$, the distribution of $\xi(s)$ has no atoms in $[-\infty, \infty]$.
(b) For every $x \in \mathbb{R} \cup \{+\infty\}$, the function $s \mapsto F_s(x)$ is usc.
Then the following statements hold:

(i) The process Z defined by $Z(s) = F_s(\xi(s))$ is a usc process with standard uniform margins.

(ii) The process $\tilde{\xi}$ defined by $\tilde{\xi}(s) = Q_s(Z(s)) = Q_s(F_s(\xi(s))$ is a usc process such that $\Pr[\tilde{\xi}(s) = \xi(s)] = 1$ for every $s \in \mathbb{D}$. In particular, the finite-dimensional distributions of $\tilde{\xi}$ and ξ are identical.

Proof. (i) By condition (a), the marginal distributions of Z are standard uniform. By condition (b), the function U defined by $(s,x) \mapsto F_s(x)$ belongs to $\mathcal{U}(\mathbb{D})$ (Definition 3.1) and we have $Z = U_*(\xi)$ with $U_* : \mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ as in Proposition 3.1. By item (iv) of that proposition, the map U_* is hypo-measurable, so that Z is a usc process too.

(ii) By Proposition A.1(v) below, we have $\Pr[\tilde{\xi}(s) = \xi(s)] = 1$ for every $s \in \mathbb{D}$. The function Q_s is the right-continuous quantile function of $\xi(s)$ and the stochastic process $(\xi(s) : s \in \mathbb{D})$ has usc trajectories. Then requirement (ii) of Proposition 4.1 is fulfilled by an application of Lemma 3.2. By item (i) of the same proposition, $\tilde{\xi}$ is a usc process. □

Remark 4.1. Although $\Pr[\tilde{\xi}(s) = \xi(s)] = 1$ for all $s \in \mathbb{D}$, it is not necessarily true that $\Pr[\tilde{\xi} = \xi] = 1$, and not even that $\tilde{\xi}$ and ξ have the same distribution as usc processes: see Examples 4.4 and 4.5. However, if $\Pr[\forall s \in \mathbb{D} : 0 < F_s(\xi(s)) < 1] = 1$ and if $Q_s(F_s(x)) = x$ for every s and every x such that $0 < F_s(x) < 1$, then clearly $\Pr[\tilde{\xi} = \xi] = 1$.

The following Lemma helps clarifying the meaning of the assumptions of Proposition 4.2.

Lemma 4.1 (Regularity of the marginal distributions w.r.t. the space variable). *If ξ is a usc process, conditions (a) and (b) in Proposition 4.2 together imply that the map $s \mapsto F_s(x) = \Pr(\xi(s) \le x)$ is continuous, for any fixed $x \in \mathbb{R}$.*

Proof. By condition (a), we have $\Pr[\xi(s) < x] = F_s(x)$ for each fixed (s, x), so that by Proposition A.1(iii), we have

$$\{s \in \mathbb{D} : x \le Q_s(p)\} = \{s \in \mathbb{D} : F_s(x) \le p\}$$

for every $x \in [-\infty, \infty]$ and $p \in [0, 1]$. Since ξ is a usc process, the function $s \mapsto Q_s(p)$ is usc for $p \in [0, 1]$ (Lemma 3.2). The set in the display is thus closed. But then the map $s \mapsto F_s(x)$ is *lower* semicontinuous, and thus, by (b), continuous. □

Remark 4.2. Condition (a) in Proposition 4.2, continuity of the marginal distributions, can perhaps be avoided using an additional randomization device, 'smearing out' the probability masses of any atoms, as in Rüschendorf (2009). However, even if this may ensure uniform margins, it may destroy upper semicontinuity, see Example 4.1. Since our interest is in extreme-value theory, in particular in max-stable distributions (Section 5), which are continuous, we do not pursue this issue further.

Remark 4.3. Without condition (b) in Proposition 4.2, the trajectories of the stochastic process $(Z(s) : s \in \mathbb{D})$ may not be usc: see Example 4.2. However, condition (b) is not necessary either: see Example 4.3.

The examples below illustrate certain aspects of Propositions 4.1 and 4.2. In the examples, we do not go into measurability issues, that is, we do not prove that the mappings ξ from the underlying probability space into $\mathrm{USC}(\mathbb{D})$ are hypo-measurable. To do so, one can for instance rely on Molchanov (2005, Example 1.2 on page 3 and Theorem 2.25 on page 37).

Example 4.1 (What may happen without (a) in Proposition 4.2)**.** Consider two independent, uniformly distributed variables X and Y on $[0, 1]$. Take $\mathbb{D} = [-1, 1]$, and define

$$\xi(s) = \begin{cases} |s|X & \text{if } -1 \le s < 0, \\ 0 & \text{if } s = 0, \\ sY & \text{if } 0 < s \le 1. \end{cases}$$

Condition (b) in Proposition 4.2 is fulfilled: the trajectories $s \mapsto F_s(x) = \Pr[\xi(s) \le x]$ are continuous for every $x \ne 0$ and still usc for $x = 0$. Still, the marginal probability integral transform produces $Z(s) = X$ for $s < 0$ and $Z(s) = Y$ for $s > 0$. Extending Z to a stochastic process on $[-1, 1]$ with usc trajectories would require $Z(0) \ge X \vee Y$. But then $Z(0)$ cannot be uniformly distributed on $[0, 1]$.

Example 4.2 (What may happen without (b) in Proposition 4.2)**.** Consider two independent, uniformly distributed variables X and Y on $[0, 1]$. On $\mathbb{D} = [0, 2]$, define

$$\xi(s) = X \vee (Y \mathbf{1}_{\{1\}}(s)) = \begin{cases} X & \text{if } s \ne 1, \\ X \vee Y & \text{if } s = 1. \end{cases}$$

Then ξ is a usc process and the distribution function, F_s, of $\xi(s)$ is given by

$$F_s(x) = \Pr[\xi(s) \le x] = \begin{cases} x & \text{if } s \neq 1, \\ x^2 & \text{if } s = 1, \end{cases}$$

where $x \in [0,1]$. As a consequence, the function $s \mapsto F_s(x)$ is lsc but not usc if $0 < x < 1$, so that condition (b) in Proposition 4.2 is violated. Reducing to standard uniform margins yields

$$Z(s) = F_s(\xi(s)) = \begin{cases} X & \text{if } s \neq 1, \\ (X \vee Y)^2 & \text{if } s = 1. \end{cases}$$

The event $\{0 < Y^2 < X < 1\}$ has positive probability, and on this event we have $(X \vee Y)^2 < X$, so that the trajectory $s \mapsto Z(s)$ is not usc. Hence, statement (i) in Proposition 4.2 fails.

Example 4.3 (Condition (b) in Proposition 4.2 is not necessary)**.** Let $D = [-1, 1]$ and let X and V be independent random variables, X standard normal and V standard uniform. Define

$$\xi(s) = \begin{cases} X & \text{if } -1 \le s \le 0, \\ X - 1 & \text{if } 0 < s < V, \\ X & \text{if } V \le s \le 1. \end{cases}$$

Let Φ be the standard normal cumulative distribution function and choose $x \in \mathbb{R}$. Then $F_s(x) = \Pr[\xi(s) \le x] = \Phi(x)$ if $s \in [-1, 0]$, while for $s \in (0, 1]$, we have

$$\begin{aligned} F_s(x) &= \Pr[s < V] \Pr[X - 1 \le x] + \Pr[V \le s] \Pr[X \le x] \\ &= (1 - s)\, \Phi(x + 1) + s\, \Phi(x). \end{aligned}$$

The function $s \mapsto F_s(x)$ is constant on $s \in [-1, 0]$ while it decreases linearly from $\Phi(x + 1)$ to $\Phi(x)$ for s from 0 to 1, the right-hand side limit at 0 being equal to $\Phi(x+1)$, which is greater than $\Phi(x)$, the value at $s = 0$ itself. Hence the function $s \mapsto F_s(x)$ is lsc but not usc, and condition (b) in Proposition 4.2 does not hold. Nevertheless, the random variables $Z(s) = F_s(\xi(s))$, $s \in \mathbb{D}$, are given as follows:

$$Z(s) = F_s(\xi(s)) = \begin{cases} \Phi(X) & \text{if } -1 \le s \le 0, \\ (1 - s)\, \Phi(X) + s\, \Phi(X - 1) & \text{if } 0 < s < V, \\ (1 - s)\, \Phi(X + 1) + s\, \Phi(X) & \text{if } V \le s \le 1. \end{cases}$$

The trajectory of Z is continuous at $s \in [-1, 1] \setminus \{V\}$ and usc at $s = V$, hence usc overall.

Example 4.4 (ξ and $\tilde{\xi}$ in Proposition 4.2 may be different in law (1))**.** Let X and Y be independent, standard uniform random variables (taking values in $[0, 1]$, to avoid trivialities). For $s \in \mathbb{D} = [0,1]$, define

$$\xi(s) = X + \mathbf{1}(Y = s) = \begin{cases} X & \text{if } s \neq Y, \\ X + 1 & \text{if } s = Y. \end{cases}$$

Since $\Pr[Y = s] = 0$ for every $s \in [0,1]$, the law of $\xi(s)$ is standard uniform and conditions (a) and (b) in Proposition 4.2 are trivially fulfilled with $F_s(x) = (x \vee 0) \wedge 1$ for $x \in [-\infty, \infty]$ and $Q_s(p) = p$ for $p \in [0,1)$ and $Q_s(1) = \infty$. We obtain

$$\tilde{\xi}(s) = Z(s) = \begin{cases} X & \text{if } s \neq Y, \\ \infty & \text{if } s = Y. \end{cases}$$

The processes $\tilde{\xi}$ and ξ have different capacity functionals and thus a different distribution as random elements in $\mathrm{USC}(\mathbb{D})$: for $K = [0,1] \times \{x\}$ with $x > 2$, we have $\Pr[\operatorname{hypo} \xi \cap K] = 0$ while $\Pr[\operatorname{hypo} \tilde{\xi} \cap K \neq \varnothing] = 1$.

Example 4.5 (ξ and $\tilde{\xi}$ in Proposition 4.2 may be different in law (2))**.** Let X and Y be independent random variables, with X uniformly distributed on $[0,1] \cup [2,3]$ and Y uniformly distributed on $[0,1]$. For $s \in \mathbb{D} = [0,1]$, define

$$\xi(s) = X \vee (1.5\, \mathbf{1}(Y = s)) = \begin{cases} X & \text{if } s \neq Y, \\ 1.5 & \text{if } s = Y \text{ and } X \leq 1. \end{cases}$$

Then $\Pr[\xi(s) = X] = 1$ for all $s \in [0,1]$, so that the marginal distribution and quantile functions F_s and Q_s do not depend on s and are equal to those of X, denoted by F_X and Q_X. The random variable $U = F_X(X)$ is uniformly distributed on $[0,1]$. Since $F_X(1.5) = F_X(1) = 0.5$, we have

$$Z(s) = F_X(\xi(s)) = \begin{cases} U & \text{if } s \neq Y, \\ 0.5 & \text{if } s = Y \text{ and } X \leq 1. \end{cases}$$

However, as $Q_X(0.5) = 2$, we obtain

$$\tilde{\xi}(s) = Q_X(Z(s)) = \begin{cases} X & \text{if } s \neq Y, \\ 2 & \text{if } s = Y \text{ and } X \leq 1. \end{cases}$$

On the event $\{X \leq 1\}$, which occurs with probability one half, the hypographs of ξ and $\tilde{\xi}$ are different.

5. Max-stable processes

We apply Propositions 4.1 and 4.2 to *max-stable* usc processes. These processes and their standardized variants are introduced in Subsection 5.2, after some preliminaries on generalized extreme-value distributions in Subsection 5.1. The main results are given Subsection 5.3, followed by some examples in Subsection 5.4.

5.1. **Generalized extreme-value distributions.** The distribution function of the generalized extreme-value (GEV) distribution with parameter vector $\theta = (\gamma, \mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R} \times (0, \infty)$ is given by

$$F(x;\theta) = \begin{cases} \exp[-\{1+\gamma(x-\mu)/\sigma\}^{-1/\gamma}] & \text{if } \gamma \neq 0 \text{ and } \sigma + \gamma(x-\mu) > 0, \\ \exp[-\exp\{-(x-\mu)/\sigma\}] & \text{if } \gamma = 0 \text{ and } x \in \mathbb{R}. \end{cases}$$

The corresponding quantile function is

$$Q(p;\theta) = \begin{cases} \mu + \sigma[\{-1/\log(p)\}^{\gamma} - 1]/\gamma & \text{if } \gamma \neq 0, \\ \mu + \sigma \log\{-1/\log(p)\} & \text{if } \gamma = 0, \end{cases} \tag{5.1}$$

for $0 < p < 1$. The support is equal to the interval $\{x \in \mathbb{R} : \sigma + \gamma(x-\mu) > 0\}$. In particular, the lower endpoint is equal to $Q(0;\theta) = -\infty$ if $\gamma \leq 0$ and $Q(0;\theta) = \mu - \sigma/\gamma$ if $\gamma > 0$.

For every n, there exist unique scalars $a_{n,\theta} \in (0,\infty)$ and $b_{n,\theta} \in \mathbb{R}$ such that the following max-stability relation holds:

$$F^n(a_{n,\theta}x + b_{n,\theta};\theta) = F(x;\theta), \qquad x \in \mathbb{R}. \tag{5.2}$$

In fact, a non-degenerate distribution is max-stable if and only if it is GEV. This property motivates the use of such distributions for modeling maxima over many variables. The location and scale sequences are given by

$$a_{n,\theta} = n^{\gamma}, \qquad\qquad b_{n,\theta} = \begin{cases} (\sigma - \gamma\mu)(n^{\gamma} - 1)/\gamma & \text{if } \gamma \neq 0, \\ \sigma \log n & \text{if } \gamma = 0. \end{cases} \tag{5.3}$$

For quantile functions, max-stability means that

$$Q(p^{1/n};\theta) = a_{n,\theta}\, Q(p;\theta) + b_{n,\theta}, \qquad p \in [0,1]. \tag{5.4}$$

Note that $Q(e^{-1};\theta) = \mu$ and thus $Q(e^{-1/n};\theta) = a_n\, \mu + b_n$.

In Sklar's theorem, the uniform distribution on $[0,1]$ plays the role of pivot. Here, it is natural to standardize to a member of the GEV family. Multiple choices are possible. We opt for the unit-Fréchet distribution, given by

$$\Phi(x) = F(x;1,1,1) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

The unit-Fréchet quantile function is given by $Q(p;1,1,1) = -1/\log(p)$ for $0 < p < 1$. If the law of X is unit-Fréchet, then the law of $Q(\Phi(X);\theta)$ is $\mathrm{GEV}(\theta)$.

For stochastic processes, the GEV parameter θ may depend on the point $s \in \mathbb{D}$. In view of the conditions in Propositions 4.1 and 4.2, the following lemma is relevant.

Lemma 5.1. *Let $\theta : \mathbb{D} \to \Theta$. The functions $s \mapsto F(x;\theta(s))$ and $s \mapsto Q(p;\theta(s))$ are usc for every $x \in \mathbb{R}$ and $p \in [0,1]$ if and only if θ is continuous.*

Proof. If θ is continuous, then the maps $s \mapsto F(x;\theta(s))$ and $s \mapsto Q(p;\theta(s))$ are continuous and thus usc.

Conversely, suppose that the functions $s \mapsto F(x;\theta(s))$ and $s \mapsto Q(p;\theta(s))$ are usc for every $x \in \mathbb{R}$ and $p \in [0,1]$. The argument is similar to the proof of Lemma 4.1. By Proposition A.1(iii), we have

$$\{s \in \mathbb{D} : x \leq Q(p;\theta(s))\} = \{s \in \mathbb{D} : F(x;\theta(s)) \leq p\}$$

for every $x \in [-\infty,\infty]$ and $p \in [0,1]$; note that GEV distributions are continuous. Since the map $s \mapsto Q(p;\theta(s))$ is usc, the above sets are closed. But then the map $s \mapsto F(x;\theta(s))$ must be lsc, and thus continuous. Finally, the map sending a GEV distribution to its parameter is continuous with respect to the weak topology (see Lemma C.1). Hence, θ is continuous. □

Remark 5.1. If $\theta : \mathbb{D} \to \Theta$ is continuous, then the normalizing functions $s \mapsto a_{n,\theta(s)}$ and $s \mapsto b_{n,\theta(s)}$ are continuous as well. Consider the map $U_n(s,x) = \{x - b_{n,\theta(s)}\}/a_{n,\theta(s)}$, for $(s,x) \in \mathbb{D} \times [-\infty,\infty]$. Clearly, $U_n \in \mathcal{U}(\mathbb{D})$. By Proposition 3.1, the transformation $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D}) : z \mapsto (z - b_{n,\theta})/a_{n,\theta}$ is then well-defined and hypo-measurable.

5.2. **Max-stable usc processes.**

Definition 5.1. A usc process ξ is called *max-stable* if, for all $n \geq 1$, there exist functions $a_n : \mathbb{D} \to (0,\infty)$ and $b_n : \mathbb{D} \to \mathbb{R}$ such that, for each vector of n independent and identically distributed (iid) usc processes $\xi_1, \ldots, \xi_n$ with the same law as ξ, we have

$$\bigvee_{i=1}^n \xi_i \stackrel{d}{=} a_n \xi + b_n. \tag{5.5}$$

A max-stable usc process ξ^* is said to be *simple* if, in addition, its marginal distributions are unit-Fréchet. In that case, the norming functions are given by $a_n(s) = n$ and $b_n(s) = 0$, i.e., in $\mathrm{USC}(\mathbb{D})$, we have

$$\bigvee_{i=1}^n \xi_i^* \stackrel{d}{=} n\xi^*$$

for iid usc processes $\xi_1^*, \ldots, \xi_n^*$ with the same law as ξ^*.

In Definition 5.1, it is implicitly understood that the functions a_n and b_n are such that the right-hand side of (5.5) still defines a usc process. If a_n is continuous and b_n is usc, then this is automatically the case; see Lemma 3.1 and Example 3.1(iii).

Equation (5.5) is not necessarily the same as saying that $(\bigvee_{i=1}^n \xi_i - b_n)/a_n$ is equal in distribution to ξ. The reason is that it is not clear that $(\bigvee_{i=1}^n \xi_i - b_n)/a_n$ is a usc process. Whether this is the case or not remains an open question.

The evaluation mappings $\mathrm{USC}(\mathbb{D}) \to [-\infty,\infty] : z \mapsto z(s)$ being measurable, equation (5.5) implies that

$$\bigvee_{i=1}^n \xi_i(s) \stackrel{d}{=} a_n(s)\,\xi(s) + b_n(s), \qquad s \in \mathbb{D}.$$

As a consequence, the marginal distribution of $\xi(s)$ is max-stable and therefore GEV with some parameter vector $\theta(s) \in \Theta$. The normalizing functions a_n and

b_n must then be of the form $a_n(s) = a_{n,\theta(s)}$ and $b_n(s) = b_{n,\theta(s)}$ as in (5.3). If $\theta : \mathbb{D} \to \Theta$ is continuous, then the normalizing functions are continuous too, and $(\bigvee_{i=1}^n \xi_i - b_n)/a_n$ is equal in distribution to ξ (Remark 5.1).

5.3. **Sklar's theorem for max-stable usc processes.** We investigate the relation between general and simple max-stable usc processes via the pointwise probability integral transform and its inverse. Max-stability of usc processes not being determined by the finite-dimensional distributions alone, it is not clear from the outset that max-stability is preserved by pointwise transformations.

Proposition 5.1 gives a necessary and sufficient condition on the GEV margins to be able to construct a general max-stable usc process starting from a simple one. Propositions 5.2 and 5.3 treat the converse question, that is, when can a max-stable usc process be first reduced to a simple one and then be reconstructed from it.

Proposition 5.1 (à la Sklar I for max-stable usc processes)**.** *Let ξ^* be a simple max-stable usc process. Let $\theta : \mathbb{D} \to \Theta$. Define a stochastic process ξ by $\xi(s) = Q(\Phi(\xi^*(s)); \theta(s))$ for $s \in \mathbb{D}$. Then the following two statements are equivalent:*

(i) ξ is a usc process with marginal distributions $\mathrm{GEV}(\theta(s))$.
(ii) For every $p \in [0, 1]$, the function $s \mapsto Q(p; \theta(s))$ is usc.

If these conditions hold, then ξ is a max-stable usc process with normalizing functions $a_n(s) = a_{n,\theta(s)}$ and $b_n(s) = b_{n,\theta(s)}$.

Proof. By Proposition 4.2(i) applied to ξ^*, the stochastic process $Z(s) = \Phi(\xi^*(s))$ induces a usc process whose margins are uniform on $[0, 1]$. The equivalence of statements (i) and (ii) then follows from Proposition 4.1.

Assume that (i) and (ii) are fulfilled. We need to show that the right-hand side in (5.5) defines a usc process and that the stated equality in distribution holds.

For positive integer n, define $U_n(s, x) = Q(\Phi(nx); \theta(s))$ for $(s, x) \in \mathbb{D}\times[-\infty, \infty]$. In view of Lemma 3.1 and Example 3.1, the map U_n belongs to $\mathcal{U}(\mathbb{D})$. Moreover, max-stability (5.4) implies

$$U_n(s, x) = Q(\Phi(x)^{1/n}; \theta(s)) = a_{n,\theta(s)}\, Q(\Phi(x); \theta(s)) + b_{n,\theta(s)}.$$

It follows that

$$a_{n,\theta(s)}\, \xi(s) + b_{n,\theta(s)} = U_n(s, \xi^*(s)).$$

By Proposition 3.1, the function $s \mapsto U_n(s, z(s))$ belongs to $\mathrm{USC}(\mathbb{D})$ for every $z \in \mathrm{USC}(\mathbb{D})$, and the map $U_{n,*}$ from $\mathrm{USC}(\mathbb{D})$ to itself sending $z \in \mathrm{USC}(\mathbb{D})$ to this function is hypo-measurable. We conclude that $a_n\, \xi + b_n$ is a usc process.

Next, we prove that ξ is max-stable. Let $\xi_1, \ldots, \xi_n$ be iid usc processes with the same law as ξ. Further, let $\xi_1^*, \ldots, \xi_n^*$ be iid usc processes with the same law as ξ^*. For every $i \in \{1, \ldots, n\}$, we have

$$\xi_i \stackrel{d}{=} \xi = U_{1,*}(\xi^*) \stackrel{d}{=} U_{1,*}(\xi_i^*).$$

The last equality in distribution comes from the hypo-measurability of $U_{1,*}$. By independence, it follows that

$$(\xi_1, \dots, \xi_n) \stackrel{d}{=} (U_{1,*}(\xi_1^*), \dots, U_{1,*}(\xi_n^*)) \,.$$

Write $(\tilde{\xi}_1, \dots, \tilde{\xi}_n) = (U_{1,*}(\xi_i^*), \dots, U_{1,*}(\xi_n^*))$. By monotonicity, we have, for $s \in \mathbb{D}$,

$$\bigvee_{i=1}^n \tilde{\xi}_i(s) = \bigvee_{i=1}^n Q\big(\Phi(\xi_i^*(s)); \theta(s)\big) = Q\big(\Phi(\bigvee_{i=1}^n \xi_i^*(s)); \theta(s)\big).$$

Since ξ^* is a simple max-stable usc process, we have $\bigvee_{i=1}^n \xi_i^* \stackrel{d}{=} n\xi^*$ in $\mathrm{USC}(\mathbb{D})$. But then also

$$\begin{aligned}
\bigvee_{i=1}^n \xi_i &\stackrel{d}{=} \bigvee_{i=1}^n \tilde{\xi}_i \\
&\stackrel{d}{=} Q\big(\Phi(n\xi^*); \theta\big) = Q\big(\Phi(\xi^*)^{1/n}; \theta\big) \\
&= a_{n,\theta}\, Q\big(\Phi(\xi^*); \theta\big) + b_{n,\theta} = a_{n,\theta}\, \xi + b_{n,\theta}.
\end{aligned}$$

□

Proposition 5.2 (à la Sklar II for usc processes with GEV margins)**.** *Let ξ be a usc process with $\mathrm{GEV}(\theta(s))$ margins for $s \in \mathbb{D}$. If $\theta : \mathbb{D} \to \Theta$ is continuous, then the following statements hold:*

(i) The process ξ^* defined $\xi^*(s) = -1/\log F(\xi(s); \theta(s))$ is a usc process with unit-Fréchet margins.

(ii) The process $\tilde{\xi}$ defined by $\tilde{\xi}(s) = Q(\Phi(\xi^*(s)); \theta(s))$ is a usc process and, with probability one,

$$\forall s \in \mathbb{D} : \tilde{\xi}(s) = \begin{cases} \xi(s) & \text{if } \xi^*(s) < \infty, \\ \infty & \text{if } \xi^*(s) = \infty. \end{cases}$$

Proof. The marginal distributions of ξ are GEV and depend continuously on s. Conditions (a) and (b) in Proposition 4.2 are therefore satisfied.

By Proposition 4.2(i), the process Z defined by $Z(s) = F(\xi(s); \theta(s))$ is a usc process with standard uniform margins. It then follows that ξ^* defined by $\xi^*(s) = -1/\log Z(s)$ is a usc process too, its margins being unit-Fréchet.

Since $0 \le Z(s) \le 1$ by construction, we have $0 \le \xi^*(s) \le \infty$ and thus $\Phi(\xi^*(s)) = Z(s)$. We find $\tilde{\xi}(s) = Q(Z(s); \theta(s)) = Q(F(\xi(s); \theta(s)); \theta(s))$. By Proposition 4.2(ii), the process $\tilde{\xi}$ is a usc process.

Recall that GEV distribution functions are continuous and strictly increasing on their domains. As a consequence, for all x such that $x \ge Q(0; \theta(s))$, we have

$$Q(F(x; \theta(s)); \theta(s)) = \begin{cases} x & \text{if } F(x; \theta(s)) < 1, \\ \infty & \text{if } F(x; \theta(s)) = 1. \end{cases}$$

Moreover, Lemma B.1 implies that $\xi \ge Q(0; \theta)$ almost surely. Since $\xi^*(s) < \infty$ if and only if $F(\xi(s); \theta(s)) < 1$, we arrive at the stated formula for $\tilde{\xi}$. □

Proposition 5.3 (à la Sklar II for max-stable processes)**.** *Let ξ be a usc process with* GEV$(\theta(s))$ *margins for $s \in \mathbb{D}$. Assume that $\sup_{s\in\mathbb{D}} F(\xi(s);\theta(s)) < 1$ with probability one and that the function $\theta : \mathbb{D} \to \Theta$ is continuous. As in Proposition 5.2, define two usc processes ξ^* and $\tilde{\xi}$ by $\xi^*(s) = -1/\log F(\xi(s);\theta(s))$ and $\tilde{\xi}(s) = Q(\Phi(\xi^*(s));\theta(s))$, for $s \in \mathbb{D}$. Then, almost surely, $\xi = \tilde{\xi}$. Furthermore, the following two statements are equivalent:*

(i) The usc process ξ is max-stable.
(ii) The usc process ξ^* is simple max-stable.

Proof. The fact that ξ^* is a usc process with unit-Fréchet margins is a consequence of the continuity of θ and Proposition 5.2(i). The hypothesis on $F(\xi(s);\theta(s))$ together with Proposition 5.2(ii) imply that $\xi = \tilde{\xi}$ almost surely.

The functions $s \mapsto a_{n,\theta(s)} \in (0,\infty)$ and $s \mapsto b_{n,\theta(s)} \in \mathbb{R}$ are continuous. The map $\mathbb{D} \times [-\infty,\infty] \to [-\infty,\infty]$ defined by $(s,x) \mapsto a_{n,\theta(s)}\, x + b_{n,\theta(s)}$ belongs to $\mathcal{U}(\mathbb{D})$. By Proposition 3.1, the map from USC$(\mathbb{D})$ to itself sending z to the function $s \mapsto a_{n,\theta(s)}\, z(s) + b_{n,\theta(s)}$ is well-defined and hypo-measurable. It follows that $a_{n,\theta}\,\xi + b_{n,\theta}$ is a usc process.

Suppose first that (ii) holds, i.e., ξ^* is simple max-stable. By Proposition 5.1, the usc process $\tilde{\xi}$ is max-stable with norming functions $s \mapsto a_{n,\theta(s)}$ and $s \mapsto b_{n,\theta(s)}$. Since ξ and $\tilde{\xi}$ are equal almost surely in USC$(\mathbb{D})$, they are also in equal in law. Statement (i) follows.

Conversely, suppose that (i) holds. Let $(\xi_i^*)_{i=1}^n$ and $(\xi_i)_{i=1}^n$ be vectors of iid usc processes with common laws equal to the ones of ξ^* and ξ, respectively. For $z \in$ USC$(\mathbb{D})$, write $-1/\log F(z,\theta) = (-1/\log F(z(s),\theta(s))_{s\in\mathbb{D}}$. The mapping $z \mapsto -1/\log F(z,\theta)$ from USC$(\mathbb{D})$ to itself is hypo-measurable, by an argument as in the proof of Proposition 5.2. For $i \in \{1,\ldots,n\}$, we have, in USC$(\mathbb{D})$,

$$\xi_i^* \stackrel{d}{=} \xi^* = -1/\log F(\xi,\theta) \stackrel{d}{=} -1/\log F(\xi_i,\theta).$$

By independence, we have thus have $(\xi_i^*)_{i=1}^n \stackrel{d}{=} (-1/\log F(\xi_i,\theta))_{i=1}^n$. Property (i) and max-stability (5.2) now say that

$$\begin{aligned}
\textstyle\bigvee_{i=1}^n \xi_i^* &\stackrel{d}{=} -1/\log F(\textstyle\bigvee_{i=1}^n \xi_i;\theta) \\
&\stackrel{d}{=} -1/\log F(a_{n,\theta}\xi + b_{n,\theta};\theta) \\
&= -1/\log\{F(\xi;\theta)^{1/n}\} \\
&= n\,[-1/\log F(\xi;\theta)] \\
&= n\xi^*.
\end{aligned}$$

□

Remark 5.2 (Regarding the finiteness of ξ^* in Proposition 5.2(ii)). Recall that usc functions reach their suprema on compacta. As a consequence, for $z \in$ USC$(\mathbb{D})$, we have $z(s) < \infty$ for all $s \in \mathbb{D}$ if and only if $\sup_{s\in\mathbb{D}} z(s) < \infty$. The event $\{\sup_{s\in\mathbb{D}} \xi^*(s) < \infty\}$ is thus the same as the event $\{\forall s \in \mathbb{D} : \xi^*(s) < \infty\}$.

Remark 5.3 (Regarding the continuity assumption on θ in Proposition 5.2). According to Lemmas 3.2 and 5.1, imposing the continuity of the GEV parameter vector $\theta(s)$ as a function of $s \in \mathbb{D}$ is equivalent to imposing the upper semicontinuity of the function $s \mapsto F(x, \theta(s))$ for each fixed $x \in \mathbb{R}$.

5.4. **Examples.** In comparison to Proposition 4.2, we have added to Proposition 5.2 the assumption that the margins be GEV. Although their distributions functions are continuous and strictly increasing on their support, this does not resolve the issues arising when the marginal distributions are not continuous in space. Example 5.1, which parallels Example 4.2, illustrates the point.

Example 5.1 (What may happen without the continuity of θ)**.** Consider two independent, unit-Fréchet distributed variables X and Y. As in Example 4.2, take $\mathbb{D} = [0, 2]$, and define

$$\xi(s) = X \vee (Y\mathbf{1}_{\{1\}}(s)) = \begin{cases} X & \text{if } s \neq 1, \\ X \vee Y & \text{if } s = 1. \end{cases}$$

Then again, ξ is a usc process. It is even a max-stable one with normalizing functions $a_n \equiv n$ and $b_n \equiv 0$. The marginal distribution functions are

$$F(x; \theta(s)) = \Pr[\xi(s) \leq x] = \begin{cases} e^{-1/x} & \text{if } s \neq 1, \\ e^{-2/x} & \text{if } s = 1, \end{cases}$$

where $x \geq 0$. Then the function $s \mapsto F_s(x)$ is lower rather than upper semicontinuous, and the marginal GEV parameter vector is

$$\theta(s) = (\gamma(s), \mu(s), \sigma(s)) = \begin{cases} (1, 1, 1) & \text{if } s \neq 1, \\ (1, 2, 2) & \text{if } s = 1, \end{cases}$$

which is not continuous as a function of s. Standardizing to Fréchet margins yields

$$\xi^*(s) = F(\xi(s); \theta(s)) = \begin{cases} e^{-1/X} & \text{if } s \neq 1, \\ e^{-2/(X \vee Y)} & \text{if } s = 1. \end{cases}$$

The event $\{0 < X < Y < 2X\}$ has positive probability, and on this event, we have $e^{-2/(X \vee Y)} = e^{-2/Y} < e^{-1/X}$. The trajectory $s \mapsto \xi^*(s)$ is therefore not usc.

To conclude, we present a construction principle for simple max-stable usc process processes. In combination with Proposition 5.1, this provides a device for the construction of max-stable usc processes with arbitrary GEV margins. The method is similar to the one proposed in Schlather (2002, Theorem 2). Proving max-stability of the usc process that we construct requires special care, since max-stability in USC($\mathbb{D}$) does not follow from max-stability of the finite-dimensional distributions.

Example 5.2. Let $Y_1 > Y_2 > Y_3 > \ldots$ denote the points of a Poisson point process on $(0, \infty)$ with intensity measure $y^{-2}\,dy$. Let $V_1, V_2, \ldots$ be iid usc processes, independent of the point process $(Y_i)_i$, with common distribution equal to the one of the usc process V. Assume that V satisfies the following properties:

- $\Pr[\inf_{s\in\mathbb{D}} V(s) \geq 0] = 1$;
- $\mathbb{E}[\sup_{s\in\mathbb{D}} V(s)] < \infty$;
- the mean function $f(s) = \mathbb{E}(V(s))$ is strictly positive and continuous on $\mathbb{D}$.

By Lemma B.3 below, $\inf_{\mathbb{D}} V$ is indeed a random variable. Note that we do *not* impose that $\inf_{\mathbb{D}} V > 0$ almost surely.

Define a stochastic process ξ on $\mathbb{D}$ by

$$\xi(s) = \sup_{i\geq 1} Y_i \frac{1}{f(s)} V_i(s), \qquad s \in \mathbb{D}.$$

We will show that ξ is 'almost surely' a simple max-stable usc process, in the following sense:

(a) With probability one, the trajectories of ξ are usc.
(b) Letting $\Omega_1 \subset \Omega$ denote a set of probability one on which the trajectories $\xi(\,\cdot\,, \omega)$ are usc, the map $\xi : \Omega_1 \to \mathrm{USC}(\mathbb{D})$ is measurable.

Consider the space of nonnegative functions

$$\mathrm{USC}(\mathbb{D})_+ = \{z \in \mathrm{USC}(\mathbb{D}) : \forall s \in \mathbb{D}, z(s) \geq 0\} = \bigcap_{s\in\mathbb{D}} \{z \in \mathrm{USC}(\mathbb{D}) :\ z(s) \geq 0\}.$$

For each fixed $s \in \mathbb{D}$, the set $\{z \in \mathrm{USC}(\mathbb{D}) : z(s) \geq 0\}$ is closed, since the corresponding set of hypographs is the complement set of $\mathcal{F}^{\{(s,0)\}}$, an open subset of $\mathcal{F}(\mathbb{D} \times \mathbb{R})$ in the Fell topology. The set $\mathrm{USC}(\mathbb{D})_+$ is thus closed in $\mathrm{USC}(\mathbb{D})$, as an intersection of closed sets.

Set $\mathbf{E} = \mathrm{USC}(\mathbb{D})_+ \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ denotes for the null function on $\mathbb{D}$. Since $\mathrm{USC}(\mathbb{D})$ is a compact space with a countable basis (Molchanov, 2005, Theorem B.2, p. 399), the space $\mathbf{E}$ is locally compact with a countable basis. Classical theory of point processes applies and it is possible to define Poisson processes on $\mathbf{E}$ by augmentation and/or continuous mappings.

Put $W_i = V_i/f$. Then for $i \in \mathbb{N}$, W_i a random element of $\mathrm{USC}(\mathbb{D})$ such that $W_i \in \mathbf{E}$ with probability one. The point process Γ defined by

$$\Gamma = \sum_{i\geq 1} \delta_{(Y_i, W_i)}.$$

is thus a Poisson process on $(0, \infty) \times \mathbf{E}$ with mean measure $\mathrm{d}\Lambda(y, w) = y^{-2}\,\mathrm{d}y \otimes \mathrm{d}P_W(w)$, where P_W is the law of W_1 (Resnick, 1987, Proposition 3.8).

The 'product' mapping $T : (0, \infty) \times \mathbf{E} \to \mathbf{E}$ defined by $T(y, w) = yw$ ($y > 0, w \in \mathbf{E}$) is measurable. Provided that the image measure $\mu = \Lambda \circ T^{-1}$ is finite on compact sets of $\mathbf{E}$, we find that the point process

$$\Pi = \sum_{i \geq 1} \delta_{(Y_i W_i)}$$

is a Poisson process with mean measure μ on $\mathbf{E}$ (Resnick, 1987, Proposition 3.7). To check finiteness of μ on compact sets of $\mathbf{E}$, we must check that for $K \subset \mathbb{D}$ and $x > 0$, writing

$$\mathcal{F}_{K \times \{x\}} = \{z \in \mathrm{USC}(\mathbb{D})_+ : \sup_{s \in K} z(s) \geq x\},$$

we have $\mu(\mathcal{F}_{K \times \{x\}}) < \infty$. Indeed, the set $\mathcal{F}_{K \times \{x\}}$ is closed in $\mathrm{USC}(\mathbb{D})$, and since $\mathrm{USC}(\mathbb{D})$ is compact, the compact sets in $\mathbf{E}$ are the closed sets F in $\mathrm{USC}(\mathbb{D})$ such that $F \subset \mathbf{E}$. Thus $\mathcal{F}_{K \times \{x\}}$ is compact. Also, any compact set in $\mathbf{E}$ must be contained in such a $\mathcal{F}_{K \times \{x\}}$. Now,

$$\begin{aligned}
\mu(\mathcal{F}_{K \times \{x\}}) &= -\Lambda\big\{(r, w) \in (0, \infty) \times \mathrm{USC}(\mathbb{D}) : r \max_{s \in K} w(s) \geq x\big\} \\
&= \mathbb{E}\left[\int_0^\infty \mathbf{1}\Big\{r \geq \inf_{s \in K} \frac{x}{W(s)}\Big\} \frac{\mathrm{d}r}{r^2}\right] \\
&= \frac{1}{x}\, \mathbb{E}\left[\sup_{s \in K_j} W(s)\right] \\
&\leq \frac{1}{x} \frac{\mathbb{E}[\sup_{\mathbb{D}} V]}{\inf_{\mathbb{D}} f},
\end{aligned}$$

which is finite by assumption on V.

To show (a), we adapt an argument of Giné et al. (1990, proof of Theorem 2.1). For fixed $K \subset \mathbb{D}$ compact and $x > 0$, we have $\mu(\mathcal{F}_{K \times \{x\}}) < \infty$ and thus $\Pi(\mathcal{F}_{K \times \{x\}}) < \infty$ almost surely. Thus, there exists a set Ω_1 of probability one, such that the following two statements hold for all $\omega \in \Omega_1$:

(i) $W_i(\,\cdot\,, \omega) \in \mathbf{E}$ for all $i \in \mathbb{N}$;
(ii) $\Pi(\mathcal{F}_{K \times \{x\}})(\omega) < \infty$ for every rational $x > 0$ and every compact rectangle $K \subset \mathbb{D}$ with rational vertices.

Take $\omega \in \Omega_1$. We need to show that for $s \in \mathbb{D}$ and $x \in \mathbb{Q}$ such that $\xi(s, \omega) < x$, we have $\limsup_{t \to s} \xi(t, \omega) \leq x$. Fix s and x as above, which implies that $x > 0$. Let $K_n \searrow \{s\}$ be a collection of compact rational rectangles as above such that s is in the interior of K_n. Then $\mathcal{F}_{\{(s,x)\}} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_{K_n \times \{x\}}$ (the inclusion '$\subset$' is immediate; the inclusion '$\supset$' is obtained by choosing for $z \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_{K_n \times \{x\}}$ and for $n \in \mathbb{N}$ a point $s_n \in K_n$ such that $z(s_n) \geq x$ and then observing that $s_n \to s$ and thus $z(s) \geq x$ by upper semicontinuity). From our choice of Ω_1, we have $\Pi(\mathcal{F}_{K_1 \times \{x\}}, \omega) < \infty$, so that the downward continuity property of the measure $\Pi(\,\cdot\,, \omega)$ applies and

$$\Pi(\mathcal{F}_{\{(s,x)\}}, \omega) = \lim_{n \to \infty} \Pi(\mathcal{F}_{K_n \times \{x\}}, \omega).$$

By our choice of s, x, and ω, the left-hand side in the display is zero. Since the sequence on the right-hand side is integer valued, there exists n_0 such that for all $n \geq n_0$, we have $\Pi(\mathcal{F}_{K_n \times \{x\}}, \omega) = 0$. This implies that for $n \geq n_0$, we have

$$\sup_{t \in K_n} Y_i(\omega) W_i(t, \omega) < x, \qquad i \in \mathbb{N}.$$

Complete the proof of (a) by noting that

$$\limsup_{t \to s} \xi(t, \omega) \leq \sup_{t \in K_{n_0}} \xi(t, \omega) = \sup_{i \in \mathbb{N}} \sup_{t \in K_{n_0}} Y_i(\omega) W_i(t, \omega) \leq x.$$

To show (b), we need to show that, for any compact $K \subset \mathbb{D} \times \mathbb{R}$, the set

$$A = \{\omega \in \Omega_1 : \operatorname{hypo} \xi(\,\cdot\,, \omega) \cap K \neq \varnothing\}$$

is a measurable subset of Ω. Notice first that $\xi(\,\cdot\,, \omega) \in \mathrm{USC}(\mathbb{D})_+$ for $\omega \in \Omega_1$; use property (i) of Ω_1. It follows that $A = \Omega_1$ as soon as K is not a subset of $\mathbb{D} \times (0, \infty)$. Assume $K \subset \mathbb{D} \times (0, \infty)$.

Then on Ω_1, we have $\operatorname{hypo} \xi \cap K \neq \varnothing$ if and only if $\Pi(\{z \in \mathbf{E} : \operatorname{hypo} z \cap K \neq \varnothing\}) \geq 1$. Now $\mathcal{F}_K = \{z \in \mathbf{E} : \operatorname{hypo} z \cap K \neq \varnothing\}$ is a measurable subset of $\mathbf{E}$, so that $X = \Pi(\mathcal{F}_K)$ is a random variable. It follows that $A = X^{-1}([1, \infty]) \cap \Omega_1$ is measurable, which shows (b).

A standard argument yields that the margins of ξ are unit-Fréchet. To show that ξ is simple max-stable, we need to show that, for independent random copies $\xi_1, \ldots, \xi_n$ of ξ, the capacity functionals of $\bigvee_{i=1}^n \xi_i$ and $n\xi$ are the same. That is, we need to show that, for every compact set in $\mathbb{D} \times \mathbb{R}$, we have

$$\Pr[\operatorname{hypo}(\textstyle\bigvee_{i=1}^n \xi) \cap K = \varnothing] = \Pr[\operatorname{hypo}(n\xi) \cap K = \varnothing]. \tag{5.6}$$

The left-hand side is equal to

$$\begin{aligned} \Pr[(\textstyle\bigcup_{i=1}^n \operatorname{hypo} \xi_i) \cap K = \varnothing] &= \Pr[\forall i = 1, \ldots, n : \operatorname{hypo} \xi_i \cap K = \varnothing] \\ &= (\Pr[\operatorname{hypo} \xi \cap K = \varnothing])^n. \end{aligned} \tag{5.7}$$

Without loss of generality, we may assume that

$$K = \textstyle\bigcup_{j=1}^p (K_j \times \{x_j\}),$$

where $p \in \mathbb{N}$ and where $K_j \subset \mathbb{D}$ is compact and x_j is real, for $j \in \{1, \ldots, p\}$. Indeed, the capacity functional of a usc process is entirely determined by its values on such compacta (Molchanov, 2005, p. 340). We may also choose $x_j > 0$, since otherwise both sides of (5.6) vanish. For such a set K, we have

$$
\begin{aligned}
&\Pr[\operatorname{hypo}\xi \cap K = \varnothing] \\
&= \Pr\Big[\Pi\big\{z \in \mathbf{E} : \exists\, 1 \le j \le p, \max_{K_j} z \ge x_j\big\} = 0\Big] \\
&= \exp\Big(-\Lambda\big\{(r,w) \in (0,\infty) \times \mathbf{E} : \exists j \le p\,,\ r \max_{s \in K_j} w(s) \ge x_j\big\}\Big) \\
&= \exp\Big(-\mathbb{E}\Big[\int_0^\infty \mathbf{1}\Big\{\exists j\ :\ r \ge \min_{s \in K_j} \frac{x_j}{W(s)}\Big\} \frac{\mathrm{d}r}{r^2}\Big]\Big) \\
&= \exp\Big(-\mathbb{E}\Big[\int_0^\infty \mathbf{1}\Big\{r \ge \min_j \min_{s \in K_j} \frac{x_j}{W(s)}\Big\} \frac{\mathrm{d}r}{r^2}\Big]\Big) \\
&= \exp\Big(-\mathbb{E}\Big[\max_j \frac{\max_{s \in K_j} W(s)}{x_j}\Big]\Big).
\end{aligned}
$$

For ξ replaced by $n\xi$, we obtain the same result, but with x_j replaced by x_j/n. In view of (5.7), the desired equality (5.6) follows.

6. Conclusion

The aim of the paper has been to extend Sklar's theorem from random vectors to usc processes. We have stated necessary and sufficient conditions to be able to construct a usc process with general margins by applying the pointwise quantile transformation to a usc process with standard uniform margins (Propositions 4.1 and 5.1). Furthermore, we have stated sufficient conditions for the pointwise probability integral transform to be possible for usc processes (Propositions 4.2, 5.2 and 5.3). These conditions imply in particular that the marginal distribution functions are continuous with respect to he space variable (Lemma 4.1). We have also provided several examples of things that can go wrong when these conditions are not satisfied. However, finding *necessary* and sufficient conditions remains an open problem.

The motivation has been to extend the margins-versus-dependence paradigm used in multivariate extreme-value theory to max-stable usc processes. The next step is to show that marginal standardization is possible in max-domain of attractions too. One question, for instance, is whether the standardized weak limit of the pointwise maxima of a sequence of usc processes is equal to the weak limit of the pointwise maxima of the sequence of standardized usc processes (Resnick, 1987, Proposition 5.10). Interesting difficulties arise: weak convergence of finite-dimensional distributions does not imply and is not implied by weak hypoconvergence; Khinchin's convergence-of-types lemma does not apply in its full generality to unions of random closed sets (Molchanov, 2005, p. 254, 'Affine normalization'). This topic will be the subject of further work.

Appendix A. Right-continuous quantile functions

The *right-continuous* quantile function, Q, of a random variable X taking values in $[-\infty, \infty]$ and with distribution function $F(x) = \Pr[X \le x]$, $x \in [-\infty, \infty]$, is defined as

$$Q(p) = \sup\{x \in \mathbb{R} : F(x) \le p\}, \qquad p \in [0, 1]. \tag{A.1}$$

By convention, $\sup \varnothing = -\infty$ and $\sup \mathbb{R} = \infty$. The fact that Q is right-continuous is stated in part (ii) of the next proposition.

Proposition A.1. *Let X be a random variable taking values in $[-\infty, \infty]$. Define $Q : [0, 1] \to [-\infty, \infty]$ as in (A.1).*

(i) For all $p \in [0, 1]$, we have $Q(p) = \sup\{x \in \mathbb{R} : \Pr(X < x) \le p\}$.
(ii) The function Q is non-decreasing and right-continuous.
(iii) For every $x \in [-\infty, \infty]$ and every $p \in [0, 1]$, we have $x \le Q(p)$ if and only if $\Pr[X < x] \le p$.
(iv) If V is uniformly distributed on $[0, 1]$, then the distribution function of $Q(V)$ is F, i.e., $Q(V)$ and X are identically distributed.
(v) If the law of X has no atoms in $[-\infty, \infty]$, then $\Pr[X = Q(F(X))] = 1$.

Proof. *(i)* Fix $p \in [0, 1]$. Since $\Pr[X < x] \le \Pr[X \le x] = F(x)$, we have

$$\begin{aligned} Q(p) &:= \sup\{x \in \mathbb{R} : F(x) \le p\} \\ &\le \sup\{x \in \mathbb{R} : \Pr[X < x] \le p\} := x_0. \end{aligned}$$

Conversely, if $Q(p) = +\infty$, there is nothing to prove. Assume then that $Q(p) < +\infty$, so that $p < 1$. Let $y > Q(p)$. We need to show that $y > x_0$ too, that is, $\Pr[X < y] > p$. But $\Pr[X < y] = \sup\{F(z) : z < y\}$, and this supremum must be larger than p, since for all $z > Q(p)$ we have $F(z) > p$.

(ii) The sets $\{x \in \mathbb{R} : F(x) \le p\}$ becoming larger with p, the function Q is non-decreasing. Next, we show that Q is right continuous at any $p \in [0, 1]$. If $Q(p) = \infty$, there is nothing to show, so suppose $Q(p) < \infty$ (in particular $p < 1$). Let $\varepsilon > 0$. Then $Q(p) + \varepsilon > Q(p)$ and thus $F(Q(p) + \varepsilon) = p + \delta > p$ for some $\delta > 0$. For $r < p + \delta$, we have $F(Q(p) + \varepsilon) > r$ too, and thus $Q(r) < Q(p) + \varepsilon$, as required.

(iii) First suppose $x < Q(p)$; we show that $\Pr[X < x] \le p$. The case $x = \infty$ is impossible, and if $x = -\infty$, then $\Pr[X < x] = 0 \le p$. So suppose that $x \in \mathbb{R}$. Using statement (i), there exists $y \in \mathbb{R}$ with $x \le y \le Q(p)$ such that $\Pr[X < y] \le p$. But then also $\Pr[X < x] \le p$.

Second suppose that $x > Q(p)$; we show that $\Pr[X < x] > p$. Clearly, we must have $Q(p) < \infty$, and so we can without loss of generality assume that x is real. But then $\Pr[X < x] > p$ by statement (i).

Finally, consider $x = Q(p)$; we show that $\Pr[X < Q(p)] \le p$. If $Q(p) = -\infty$, then $\Pr[X < Q(p)] = 0 \le p$. If $Q(p) > -\infty$, then $\Pr[X < y] \le p$ for all $y < Q(p)$, and thus $\Pr[X < Q(p)] = \sup_{y<Q(p)} \Pr[X < y] \le p$ too.

(iv) Without loss of generality, assume that $0 \le V \le 1$ (if not, then replace V by $(V \vee 0) \wedge 1$, which is almost surely equal to V). Let $x \in [-\infty, \infty]$. By statement (iii), we have $x \le Q(V)$ if and only if $\Pr[X < x] \le V$. As a consequence, $\Pr[x \le Q(V)] = \Pr[\Pr[X < x] \le V] = 1 - \Pr[X < x] = \Pr[x \le X]$. We conclude that $Q(V)$ and X are identically distributed and thus that $Q(V)$ has distribution function F too.

(v) By definition, $x \le Q(F(x))$ for every $x \in \mathbb{R}$. For $x = -\infty$ and $x = +\infty$, the same inequality is trivially fulfilled too (recall that $F(\infty) = 1$ and $Q(1) = \infty$). As a consequence, $X \le Q(F(X))$.

Conversely, let $\mathcal{P}$ be the collection of $p \in [0, 1]$ such that the set $\{x \in \mathbb{R} : F(x) = p\}$ has positive Lebesgue measure. These sets being disjoint for distinct p, the set $\mathcal{P}$ is at most countably infinite. If $x < Q(F(x))$, then there exists $y > x$ such that $F(x) = F(y)$ and thus $F(x) \in \mathcal{P}$. However, the law of $F(X)$ is standard uniform, so that $\Pr[F(X) \in \mathcal{P}] = 0$. Hence $X = Q(F(X))$ almost surely. □

Proposition A.2. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables defined on the same probability space and taking values in $[-\infty, \infty]$. Let Q_n and Q be the right-continuous quantile functions (A.1) of X_n and $\limsup_{n\to\infty} X_n$, respectively. Then*

$$Q(p) \ge \limsup_{n\to\infty} Q_n(p), \qquad p \in [0, 1].$$

Proof. If $Q(p) = \infty$, there is nothing to show, so suppose $Q(p) < \infty$. Let $y > Q(p)$; we will show that $y \ge \limsup_{n\to\infty} Q_n(p)$. This being true for all $y > Q(p)$, we will have proved the proposition.

By statement (i) of Proposition A.1, we have $\Pr[\limsup_{n\to\infty} X_n < y] > p$. By Fatou's lemma, there exists a positive integer $n(y)$ such that $\Pr[X_n < y] > p$ for all integer $n \ge n(y)$. But for such n, we have $y > Q_n(p)$ too. Hence, $y \ge \limsup_{n\to\infty} Q_n(p)$, as required. □

Appendix B. Lower bounds of usc processes

Lemma B.1. *If ξ is a usc process, then the function $\ell(s) = \sup\{x \in \mathbb{R} : \Pr[\xi(s) < x] = 0\}$, for $s \in \mathbb{D}$, is usc. Moreover, $\xi \vee \ell$ is a usc process too, and we have $\Pr[\xi \ge \ell] = \Pr[\xi = \xi \vee \ell] = 1$.*

Proof. The function ℓ is equal to the function $s \mapsto Q_s(0)$, with Q_s the right-continuous quantile function (A.1) of ξ_s. By Lemma 3.2, the function ℓ is usc, and by Example 3.1, the map $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D}) : z \mapsto z \vee \ell$ is hypo-measurable, so that $\xi \vee \ell$ is a usc process too.

By Lemma B.2, there exists a countable subset, $\mathbb{Q}_\ell$, of $\mathbb{D}$ with the following property: for all $x \in \mathrm{USC}(\mathbb{D})$, we have $x(s) \ge \ell(s)$ for all $s \in \mathbb{D}$ if and only if $x(t) \ge \ell(t)$ for all $t \in \mathbb{Q}_\ell$. Since $\ell(t) = Q_t(0)$, we have $\Pr[\xi(t) \ge \ell(t)] = 1$ for all $t \in \mathbb{Q}_\ell$; see Proposition A.1(iii). Since $\mathbb{Q}_\ell$ is countable, also $\Pr[\forall t \in \mathbb{Q}_\ell : \xi(t) \ge \ell(t)] = 1$. By the property of $\mathbb{Q}_\ell$ mentioned earlier, the event $\{\forall t \in \mathbb{Q}_\ell : \xi(t) \ge \ell(t)\}$ is equal to both $\{\xi \ge \ell\}$ and $\{\xi = \xi \vee \ell\}$. □

Lemma B.2. *For every $z \in \mathrm{USC}(\mathbb{D})$, there exists a countable set $\mathbb{Q}_z \subset \mathbb{D}$ such that*

$$z(s) = \inf_{\varepsilon>0} \sup_{t \in \mathbb{Q}_z : d(s,t) \le \varepsilon} z(t), \qquad s \in \mathbb{D}. \tag{B.1}$$

In particular, for every $x \in \mathrm{USC}(\mathbb{D})$, we have $x(s) \ge z(s)$ for all $s \in \mathbb{D}$ if and only if $x(t) \ge z(t)$ for all $t \in \mathbb{Q}_z$.

Proof. The set $\operatorname{hypo} z$ is a subset of the metrizable separable space $\mathbb{D} \times \mathbb{R}$. Hence, it is separable too. Let $\mathbb{Q}$ be a countable, dense subset of $\operatorname{hypo} z$. Let $\mathbb{Q}_z$ be the set of $t \in \mathbb{D}$ such that $(t, x) \in \mathbb{Q}$ for some $x \in \mathbb{R}$. Then $\mathbb{Q}_z$ is a countable subset of $\mathbb{D}$.

Let $y(s)$ denote the right-hand side of (B.1). Since z is usc, we have $z(s) \ge y(s)$ for all $s \in \mathbb{D}$.

Conversely, let $s \in \mathbb{D}$. If $z(s) = -\infty$, then trivially $y(s) \ge z(s)$. Suppose $z(s) > -\infty$. Let $-\infty < \alpha < z(s)$, so that $(s, \alpha) \in \operatorname{hypo} z$. Find a sequence $(t_n, \alpha_n) \in \mathbb{Q}$ such that $(t_n, \alpha_n) \to (s, \alpha)$ as $n \to \infty$. Then $(t_n, \alpha_n) \in \operatorname{hypo} z$ and thus $z(t_n) \ge \alpha_n$ for all n. Moreover, $t_n \to s$ and $\alpha_n \to \alpha$ as $n \to \infty$. It follows that $y(s) \ge \alpha$. Since this is true for all $\alpha < z(s)$, we find $y(s) \ge z(s)$.

We prove the last statement. Let $x \in \mathrm{USC}(\mathbb{D})$ and suppose that $x(t) \ge z(t)$ for all $t \in \mathbb{Q}_z$. The function x is equal to its own usc hull, i.e.,

$$x(s) = \inf_{\varepsilon>0} \sup_{t \in \mathbb{D} : d(s,t) \le \varepsilon} x(t), \qquad s \in \mathbb{D}.$$

Combine this formula together with (B.1) to see that $x(s) \ge z(s)$ for all $s \in \mathbb{D}$. □

Lemma B.3. *If ξ is a usc process, then $\inf_{s \in F} \xi(s)$ is a random variable for any closed set $F \subset \mathbb{D}$.*

Proof. Let $\mathbb{Q}$ be a countable, dense subset of F. Since every $\xi(s)$ is a random variable, it suffices to show that $\inf_{s \in F} \xi(s) = \inf_{s \in \mathbb{Q}} \xi(s)$. The inequality '$\le$' is trivial. To see the other inequality, suppose that x is such that $\xi(s) \ge x$ for all $s \in \mathbb{Q}$. Then $(s, x) \in \operatorname{hypo} \xi$ for all $s \in \mathbb{Q}$, and thus $(s, x) \in \operatorname{hypo} \xi$ for all $s \in F$, since $\operatorname{hypo} \xi$ is closed. It follows that $\xi(s) \ge x$ for all $s \in F$. □

Appendix C. Continuity of the GEV parameter

Recall the GEV distributions from Subsection 5.1.

Lemma C.1. *Let $F_n = F(\,\cdot\,, \theta_n)$, $n \ge 0$, be GEV distribution functions with associated GEV parameters $\theta_n = (\mu_n, \sigma_n, \gamma_n)$. If $(F_n)_n$ converges weakly to F_0, then also $\lim_{n\to\infty} \theta_n = \theta_0$ in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$.*

Proof. By continuity of the GEV distribution, weak convergence is the same as pointwise convergence, and thus $\lim_{n\to\infty} F_n(x) = F_0(x)$ for all $x \in \mathbb{R}$.

Recall the expression (5.1) of the quantile function $Q(\,\cdot\,; \theta)$. Pointwise convergence of monotone functions implies pointwise convergence of their inverses at continuity points of the limit (Resnick, 1987, Chapter 0). Setting $p = e^{-1}$, we obtain

$$\mu_n = Q(e^{-1}; \theta_n) \to Q(e^{-1}; \theta_0) = \mu_0, \qquad n \to \infty.$$

As a consequence, for $p \in (0, 1)$,

$$\begin{aligned} Q(p; \gamma_n, 0, \sigma_n) &= Q(p; \theta_n) - \mu_n \\ &\to Q(p; \theta_0) - \mu_0 = Q(p; \gamma_0, 0, \sigma_0), \qquad n \to \infty. \end{aligned} \tag{C.1}$$

This implies that for $x, y > 0$ such that $y \neq 1$,

$$\lim_{n\to\infty} \frac{x^{\gamma_n} - 1}{y^{\gamma_n} - 1} = \frac{x^{\gamma_0} - 1}{y^{\gamma_0} - 1}.$$

For $\gamma_n = 0$, the above expressions are to be understood as $\log(x)/\log(y)$. A subsequence argument then yields that $(\gamma_n)_n$ must be bounded, and a second subsequence argument confirms that $\gamma_n \to \gamma_0$ as $n \to \infty$. Combine this convergence relation with (C.1) and use the identity $Q_n(p; \gamma_n, 0, \sigma_n) = \sigma_n \, Q(p; \gamma_n, 0, 1)$ to conclude that $\sigma_n \to \sigma_0$ as $n \to \infty$. □

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